§6.2 Happel's Theorem.

Theorem 6.2.1 Let $(\mathcal{B}, \mathcal{S})$ be a Frobenius cat. Then $(\underline{\mathcal{B}}, T, \mathcal{E})$ is a Δ-cat.

proof of (tr4).

Let.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "i"] \arrow[d, "v"] \arrow[dr, phantom, "①"] & Z' \arrow[r, "i'"] \arrow[d, dashed, "\exists f"] \arrow[dr, phantom, "②"] & TX \arrow[d, equal] & \in \Omega \\
X \arrow[r, "vu"] & Z \arrow[r, "k"] \arrow[d, "j"] \arrow[dr, phantom, "③"] & Y' \arrow[r, "k'"] \arrow[d, dashed, "\exists g"] \arrow[dr, phantom, "④"] & TX \arrow[d, "Tu"] & \in \Omega \\
 & X' \arrow[r, equal] \arrow[d, "j'"] \arrow[dr, phantom, "⑤"] & X' \arrow[r] \arrow[d, dashed, "(Ti)j'"] & TY & \\
 & TY \arrow[r, "Ti"] & TZ' & & \\
 & \Omega & \mathcal{E} & &
\end{tikzcd}$$

step 1: Construct $f: Z' \longrightarrow Y'$.

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "m_X"] \arrow[d, "u"] & I(X) \arrow[r, "p_X"] \arrow[d, "\tilde{u}"] \arrow[dd, bend left, "\widetilde{vu}"] & TX \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r, "i"] \arrow[dr, bend right, "kv"] & Z' = C_u \arrow[r, "i'"] \arrow[d, dashed, "\exists f"] & TX \arrow[r] & 0 \\
 & & Y' & &
\end{tikzcd}$$

(6.5)

$(i, -\tilde{u}): Y \oplus I(X) \longrightarrow Z'$.

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "m_Y"] \arrow[d, "v"] & I(Y) \arrow[r, "p_Y"] \arrow[d, "\tilde{v}"] & TY \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Z \arrow[r, "j"] & X' = C_v \arrow[r, "j'"] & TY \arrow[r] & 0
\end{tikzcd}$$

(6.6)

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "m_X"] \arrow[d, "vu"] & I(X) \arrow[r, "p_X"] \arrow[d, "\widetilde{vu}"] \arrow[dd, bend left, "?"] & TX \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Z \arrow[r, "k"] \arrow[dr, bend right, "j"] & Y' = C_{vu} \arrow[r, "k'"] & TX \arrow[r] & 0 \\
 & & X' & &
\end{tikzcd}$$

(6.7).

$\widetilde{vu} \circ m_X = k v u$.

$\exists f: Z' \to Y'$, s.t. $fi = kv$ ①, $f\tilde{u} = \widetilde{vu}$ (6.8)

$k'f \overset{?}{=} i'$ $\quad k'f(i, -\tilde{u}) \overset{(6.8)}{=} (k'kv, -k'\widetilde{vu}) \overset{(6.7)}{=} (0, -p_X) \overset{(6.5)}{=} i'(i, -\tilde{u})$

$\Rightarrow k'f = i'$ ②

Step2: Construct $g' : Y' \longrightarrow \widetilde{X'} \xrightarrow{\ r\ } X'$

$m_{Z'} i : Y \hookrightarrow I(Z')$ where $m_{Z'} : Z' \hookrightarrow I(Z')$

$$0 \to Y \xrightarrow{m_{Z'} i} \underbrace{\underline{I(Z')}}_{S\text{-inj.}} \xrightarrow{\pi} \overset{\mathrm{cok}(m_{Z'} i)}{\overset{\parallel}{M}} \longrightarrow 0 \ \in S.$$

Consider:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Y \arrow[r, "i"] \arrow[d, equal] & Z' \arrow[r, "j'"] \arrow[d, "m_{Z'}"] \arrow[dr, phantom, "\text{p.o.}"] & TX \arrow[r] \arrow[d, dashed, "\exists !", "\text{p.B.}"'] & 0 \\
0 \arrow[r] & Y \arrow[r, "m_{Z'} i"] & I(Z') \arrow[r, "\pi"] \arrow[d, "p_{Z'}"] & M \arrow[r] \arrow[d] & 0 \\
 & & TZ' \arrow[r, equal] \arrow[d] & TZ' \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

$\Rightarrow M \in \mathcal{B}$.

$$0 \to Y \xrightarrow{m_Y} \underbrace{\underline{I(Y)}}_{S\text{-inj.}} \xrightarrow{p_Y} TY \ \in S.$$

we have comm. diag:

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "m_Y"] \arrow[d, equal] & I(Y) \arrow[r, "p_Y"] \arrow[d, "\alpha = I(i)"] & TY \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & Y \arrow[r, "m_{Z'} i"] \arrow[d, equal] & I(Z') \arrow[r, "\pi"] \arrow[d, "\alpha'"] & M \arrow[r] \arrow[d, "\beta'"] & 0 \\
0 \arrow[r] & Y \arrow[r, "m_Y"] & I(Y) \arrow[r, "p_Y"] & TY \arrow[r] & 0
\end{tikzcd} \qquad (6.11)$$

By Lem 6.1.3. we have $\underline{\beta'\beta} = \mathrm{id}_{TY}$. $\underline{\beta\beta'} = \mathrm{id}_M$ in $\underline{\mathcal{B}}$.

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "m_Y"] \arrow[d, "i"] & I(Y) \arrow[r, "p_Y"] \arrow[d, "I(i)"] & TY \arrow[r] \arrow[d, "Ti"] & 0 \\
0 \arrow[r] & Z' \arrow[r, "m_{Z'}"] & I(Z') \arrow[r, "p_{Z'}"] & TZ' \arrow[r] & 0
\end{tikzcd} \qquad (6.12)$$

we choose $\alpha = I(i)$.

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "m_{Z'} i"] \arrow[d, "\vartheta"] \arrow[dr, phantom, "\text{p.o.}"] & I(Z') \arrow[r, "\pi"] \arrow[d, "i_\vartheta"] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Z \arrow[r, "\tilde{j}"] & \widetilde{X'} = C_\vartheta \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
 & & I(Z') \\
Y \arrow[urr, bend left] \arrow[rr] \arrow[d, "\vartheta"] & & I(Y) \arrow[d] \\
Z \arrow[rr] & & X'
\end{tikzcd}$$

Lem 6.2.2 (1). $\Rightarrow$ there are $r : X' \longrightarrow \widetilde{X'}$, $r' : \widetilde{X'} \longrightarrow X'$ s.t.

$$\begin{cases} \underline{r'r} = \mathrm{Id}_{\widetilde{X}}, & \underline{rr'} = \mathrm{Id}_{X'} \\ r i_\vartheta = i_\vartheta \alpha = i_\vartheta I(i), & rj = \tilde{j} \\ r' i_\vartheta = i_\vartheta \alpha', & r'j = j \end{cases} \qquad (6.14).$$

Consider

$$\begin{tikzcd}
X \arrow[r, "m_X"] \arrow[d, "vu"'] & I(X) \arrow[d, "\iota_{vu}"] \arrow[ddr, bend left, "\iota_{v'} m_{Z'} \iota_u"] & \\
Z \arrow[r, "k"] \arrow[drr, bend right, "\tilde{j}"'] & Y' \arrow[dr, dashed, "g'"] & \\
 & & \widetilde{X'}
\end{tikzcd}$$

$\iota_{v'} m_{Z'} \iota_u : I(X) \to Z' \to I(Z') \to \widetilde{X'}$

s.t. $g'k = \tilde{j}$, $g' \iota_{vu} = \iota_{v'} m_{Z'} \iota_u$

(6.15).

$$\tilde{j} vu \overset{(6.13)}{=} \iota_{v'} m_{Z'} \iota_u \overset{(6.5)}{=} \iota_{v'} m_{Z'} \iota_u m_X.$$

Step 3: (1) $Z' \to Y' \to \widetilde{X'} \to TZ' \in \Omega.$

$$\begin{tikzcd}
0 \arrow[r] & Z' \arrow[r, "m_{Z'}"] \arrow[d, "f"'] \arrow[dr, phantom, "(*)"] & I(Z') \arrow[r] \arrow[d, "\iota_{v'}"] \arrow[dd, bend left, "b"] & TZ' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y' \arrow[r, "g'"] \arrow[dr, bend right, "a"'] & \widetilde{X'} \arrow[r] \arrow[d, dashed, "s"] & TZ' \arrow[r] & 0 \\
 & & D & &
\end{tikzcd}$$

① comm: $g'f(\iota, -\iota_v) = \iota_{v'} m_{Z'} (\iota, -\iota_u).$

$\Rightarrow$ (*) comm.

② $\forall D$, if $af = b m_{Z'}$ find $s : \widetilde{X'} \to D$.

By (6.13).

$$\begin{tikzcd}
Y \arrow[r, "m_Z \iota"] \arrow[d, "v"'] \arrow[dr, phantom, "P.O."] & I(Z') \arrow[d, "\iota_{v'}"] \arrow[ddr, bend left, "b"] & \\
Z \arrow[r, "\tilde{j}"] \arrow[drr, bend right, "ak"'] & \widetilde{X'} \arrow[dr, dashed, "\exists ! s"] & \\
 & & D
\end{tikzcd}$$

$b m_Z \iota = a f \iota \overset{(6.8)}{=} a k v.$

· $sg' \overset{?}{=} a.$ $(k, -\iota_{vu}) : Z \oplus I(X) \to Y'.$

$sg'(k, -\iota_{vu}) = \cdots\cdots = a(k, -\iota_{vu}) \Rightarrow sg' = a.$

$\Rightarrow Z' \xrightarrow{f} Y \xrightarrow{g'} \widetilde{X'} \xrightarrow{w} TZ' \in \Omega.$

(2) $Z' \xrightarrow{f} Y' \xrightarrow{r'g'} X' \to TZ' \in \Xi.$

Let $g = r'g' : Y' \to X'$. By (6.15) & (6.14), we have :

$$gk = j \quad \text{and} \quad g\, i_{0u} = i_0 \alpha' m_{Z'} i_u \qquad (6.17)$$

③

$$\begin{tikzcd}
Z' \arrow[r] \arrow[d, equal] & Y' \arrow[r] \arrow[d, equal] & \tilde{X}' \arrow[r] \arrow[d, "\gamma'"'] & TZ' \arrow[d, equal] \\
Z' \arrow[r, "f"] & Y' \arrow[r, "g"] & X' \arrow[r, "wr"] & TZ'
\end{tikzcd} \in \mathcal{E}.$$

Step 4: claim ⑤ $wr = (Ti)j'$ $\qquad (6.18)$

consider $(j, -i_0) : Z \oplus I(Y) \to X'$ surj.

$wr(j, -i_0) = \cdots = (Ti)j'(j, -i_0)$.

$\Rightarrow \quad wr = (Ti)j'$

step 5. verify ④ : $j'g = (Tu)k'$ $: Y' \to$

$(k, -i_{0u}) : Z \oplus I(X) \to Y'$.

By (6.5), (6.10) and (6.11)

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "m_X"] \arrow[d, "u"'] & I(X) \arrow[r, "p_X"] \arrow[d, "i_u"] & TX \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r, "i"] \arrow[d, equal] & Z' \arrow[r, "i'"] \arrow[d, "m_{Z'}"'] & TX \arrow[r] \arrow[d, "t"] & 0 \\
0 \arrow[r] & Y \arrow[r, "m_{Z'} i"] \arrow[d, equal] & I(Z') \arrow[r, "\pi"] \arrow[d, "\alpha'"'] & M \arrow[r] \arrow[d, "\beta'"'] & 0 \\
0 \arrow[r] & Y \arrow[r, "m_Y"] & I(Y) \arrow[r, "p_Y"] & TY \arrow[r] & 0
\end{tikzcd}$$

$$\Rightarrow \quad \begin{tikzcd}
0 \arrow[r] & X \arrow[r, "m_X"] \arrow[d, "u"'] & I(X) \arrow[r, "p_X"] \arrow[d, "\alpha' m_{Z'} i_u"'] \arrow[d, shift left=2, "I(u)"] & TX \arrow[r] \arrow[d, "\beta' t"'] \arrow[d, shift left=2, "Tu"] & 0 \\
0 \arrow[r] & Y \arrow[r, "m_Y"] & I(Y) \arrow[r, "p_Y"] & TY \arrow[r] & 0
\end{tikzcd}$$

$\Rightarrow (\alpha' m_{Z'} i_u - I(u))\, m_X = 0.$

there is $a : TX \to I(Y)$ s.t.

$$\alpha' m_{Z'} i_u - I(u) = a p_X \qquad (6.19)$$

$$j'g(k, -i_{Y}u) \overset{(6.17)}{=} j'(j, -i_Y \alpha' m_Z' i_u) \overset{(6.6)}{=} (0, -P_Y \alpha' m_Z' i_u)$$

$$(Tu)k'(k, -i_Y u) \overset{(6.17)}{=} (0, -T(u) P_X) = (0, -P_Y I(u)).$$

Hence $\underline{(j'g - (Tu)k')}\,\cancel{(k, -i_Y u)}$

$$= (0, -P_Y(\alpha' m_Z' i_u - I(u))) \overset{(6.19)}{=} (0, -P_Y a P_X)$$

$$= \underline{P_Y a k'}\,\cancel{(k, -i_Y u)}.$$

$\underline{j'g} = (Tu)\underline{k'}$.

#.

§6.3. Another explanation of d.△ in $\underline{\mathcal{B}}$. Allen Hatch

prop 6.3.1 Let $(\mathcal{B}, \mathcal{S})$ be a Frobenius cat whose stable cat. is $\underline{\mathcal{B}}$. Then the d-△ in $\underline{\mathcal{B}}$ are all induced by exact seq in $\mathcal{B}$.

(1) Suppose $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0 \in \mathcal{S}$. Then $X \xrightarrow{\underline{u}} Y \xrightarrow{\underline{v}} Z \xrightarrow{-\underline{w}} TX \in \mathcal{E}$.

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, dashed, "\sigma"] & Z \arrow[r] \arrow[d, dashed, "w"] & 0 \\
0 \arrow[r] & X \arrow[r, "m_X"] & I(X) \arrow[r, "P_X"] & TX \arrow[r] & 0.
\end{tikzcd} \qquad (6.20).$$

w', w $\quad \triangle \cong \triangle'$.

(2) Let $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{-w'} TX' \in \mathcal{E}$, then $\exists$ s.e.s in $\mathcal{S}$: $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ inducing a d-△. in $\underline{\mathcal{B}}$: $X \xrightarrow{\underline{u}} Y \xrightarrow{\underline{v}} Z \xrightarrow{-\underline{w}} TX$ is isom. to $X' \to Y' \to Z' \to TX'$.

prop. "⇒"

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "u"] \arrow[d, "m_X"'] \arrow[dr, phantom, "\text{P.O. } \text{P.B.}"] & Y \arrow[r, "v"] \arrow[d, "\binom{\sigma}{v}"] \arrow[dr, phantom, "\wr"] & Z \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I(X) \arrow[r, "\binom{1}{0}"] & I(X) \oplus Z \arrow[r, "(0{,}1)"] & Z \arrow[r] & 0
\end{tikzcd}$$

consider

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "m_X"] \arrow[d, "u"'] \arrow[dr, phantom, "\text{p.o.}"] & I(X) \arrow[r, "p_X"] \arrow[d, "\binom{1}{0}"] & TX \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r, "\binom{\sigma}{v}"'] & I(X) \oplus Z \arrow[r, "(p_X{,}-w)"] & TX \arrow[r] & 0
\end{tikzcd}$$

$\Rightarrow$ $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{-w} TX \in \mathcal{E}$.

"⇐" $\exists$ standard $\triangle$: $(X \xrightarrow{u} Y \xrightarrow{v} C_u \xrightarrow{w} TX)$

$\cong X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{-w'} TX'$

By

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "m_X"] \arrow[d, "u"'] \arrow[dr, phantom, "\text{P.O.}"] & I(X) \arrow[r, "p_X"] \arrow[d, "\tilde{u}"] & TX \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r, "v"] & C_u \arrow[r, "w"] & TX \arrow[r] & 0
\end{tikzcd}$$

(*)

$\Rightarrow$ s.e.s in S :

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\binom{m_X}{u}"] \arrow[d, equal] & I(X) \oplus Y \arrow[r, "(v{,}-\tilde{u})"] \arrow[d, "(0{,}1)"] \arrow[dr, phantom, "\wr"] & C_u \arrow[r] \arrow[d, dotted, "\exists -w"] & 0 \\
0 \arrow[r] & X \arrow[r, "m_X"] & I(X) \arrow[r, "p_X"] & TX \arrow[r] & 0
\end{tikzcd}$$

$\Rightarrow$ $X \xrightarrow{u} Y \xrightarrow{v} C_u \xrightarrow{w} TX$

#