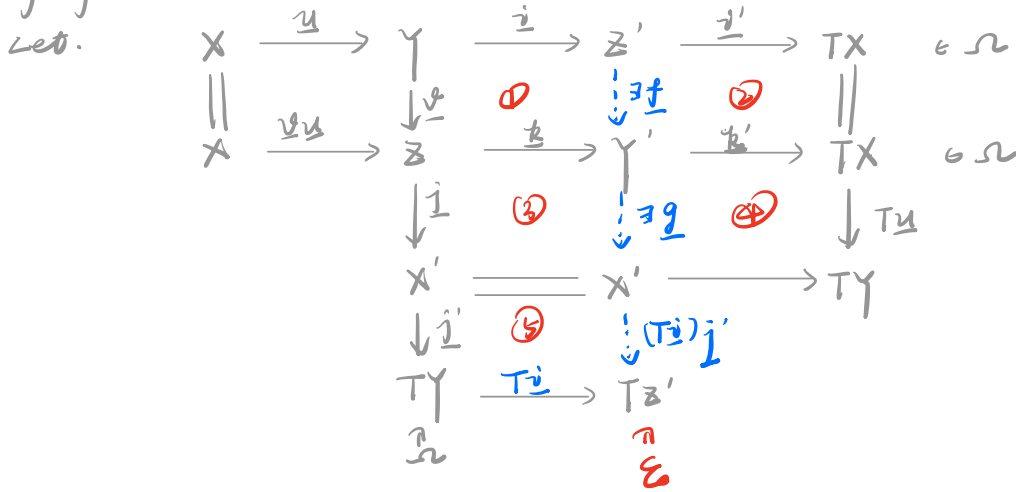


3.6.2 Happel's Theorem.

Theorem 6.2.1 Let $(\mathcal{B}, \mathcal{S})$ be a Frobenius cat. Then $(\underline{\mathcal{B}}, T, \mathcal{E})$ is a Δ -cat.

proof of (tr 4).



step 1: Construct $f: Z' \rightarrow Y'$.

$$\begin{array}{ccccccc}
 0 & \rightarrow & X & \xrightarrow{m_X} & I(X) & \xrightarrow{P_X} & TX \rightarrow 0 \\
 & & \downarrow u & & \downarrow i'u & \textcircled{1} & \parallel \\
 0 & \rightarrow & Y & \xrightarrow{i} & Z' = \text{Co} & \xrightarrow{i'} & TX \rightarrow 0 \\
 & & & & \downarrow \exists f & & \\
 & & & & Y' & &
 \end{array} \quad (6.5)$$

$(i, -i'u) = Y \oplus I(X) \rightarrow Z'$

$$\begin{array}{ccccccc}
 0 & \rightarrow & Y & \xrightarrow{m_Y} & I(Y) & \xrightarrow{P_Y} & TY \rightarrow 0 \\
 & & \downarrow v & & \downarrow i'v & \parallel & \\
 0 & \rightarrow & Z & \xrightarrow{j} & X' = \text{Co} & \xrightarrow{j'} & TY \rightarrow 0 \\
 & & & & & &
 \end{array} \quad (6.6)$$

$$\begin{array}{ccccccc}
 0 & \rightarrow & X & \xrightarrow{m_X} & I(X) & \xrightarrow{P_X} & TX \rightarrow 0 \\
 & & \downarrow v \circ u & & \downarrow i'v \circ u & \textcircled{?} & \parallel \\
 0 & \rightarrow & Z & \xrightarrow{k} & Y' = \text{Co} & \xrightarrow{k'} & TX \rightarrow 0 \\
 & & & & \downarrow \exists j & & \\
 & & & & X' & &
 \end{array} \quad (6.7)$$

$i'v \circ m_X = k \circ u$

$\exists f: Z' \rightarrow Y'$, s.t. $f \circ i = k \circ v$ $\textcircled{1}$, $f \circ i'u = v \circ u$ (6.8)

$k' \circ f \stackrel{?}{=} i'$ $k' \circ f \circ (i, -i'u) \stackrel{(6.8)}{=} (k' \circ k \circ v, -k' \circ i' \circ v \circ u) \stackrel{(6.7)}{=} (0, -P_X) \stackrel{(6.5)}{=} i' \circ (i, -i'u)$

$\Rightarrow \underline{k' \circ f = i'} \textcircled{2}$

step 2: Construct $g' : Y' \rightarrow \tilde{X}' \xrightarrow{r} X'$

$$m_{Z'} \circ i : Y \hookrightarrow I(Z') \text{ where } m_{Z'} : Z' \hookrightarrow I(Z')$$

$$0 \rightarrow Y \xrightarrow{m_{Z'} \circ i} I(Z') \xrightarrow{\pi} \overset{\text{coker}(m_{Z'} \circ i)}{M} \rightarrow 0 \in \mathcal{S}.$$

Consider:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & Y & \xrightarrow{j} & Z' & \xrightarrow{j'} & TX \longrightarrow 0 \\
 & & \parallel & & \downarrow m_{Z'} & \text{P.O.} & \downarrow \exists \theta \\
 0 & \longrightarrow & Y & \xrightarrow{m_{Z'} \circ i} & I(Z') & \xrightarrow{\pi} & M \longrightarrow 0 \\
 & & & & \downarrow \beta' & & \downarrow \\
 & & & & TZ' & = & TZ' \\
 & & & & \downarrow & & \downarrow
 \end{array} \Rightarrow M \in \mathcal{B}.$$

$$0 \rightarrow Y \xrightarrow{m_Z} I(Y) \xrightarrow{P_Y} TY \in \mathcal{S}.$$

we have comm. diag:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & Y & \xrightarrow{m_Y} & I(Y) & \xrightarrow{P_Y} & TY \longrightarrow 0 \\
 & & \parallel & & \downarrow \alpha = I(i) & & \downarrow \beta \\
 0 & \longrightarrow & Y & \xrightarrow{m_{Z'} \circ i} & I(Z') & \xrightarrow{\pi} & M \longrightarrow 0 \\
 & & \parallel & & \downarrow \alpha' & & \downarrow \beta' \\
 0 & \longrightarrow & Y & \xrightarrow{m_Y} & I(Y) & \xrightarrow{P_Y} & TY \longrightarrow 0
 \end{array} \tag{6.11}$$

By lem 6.1.b. we have $\beta' \beta = \text{id}_{TY}$, $\beta \beta' = \text{id}_M$ so β .

$$\begin{array}{ccccccc}
 0 & \longrightarrow & Y & \xrightarrow{m_Y} & I(Y) & \xrightarrow{P_Y} & TY \longrightarrow 0 \\
 & & \downarrow i & & \downarrow I(i) & & \downarrow T i \\
 0 & \longrightarrow & Z' & \xrightarrow{m_{Z'}} & I(Z') & \xrightarrow{P_{Z'}} & TZ' \longrightarrow 0
 \end{array} \tag{6.12}$$

we choose $\alpha = I(i)$.

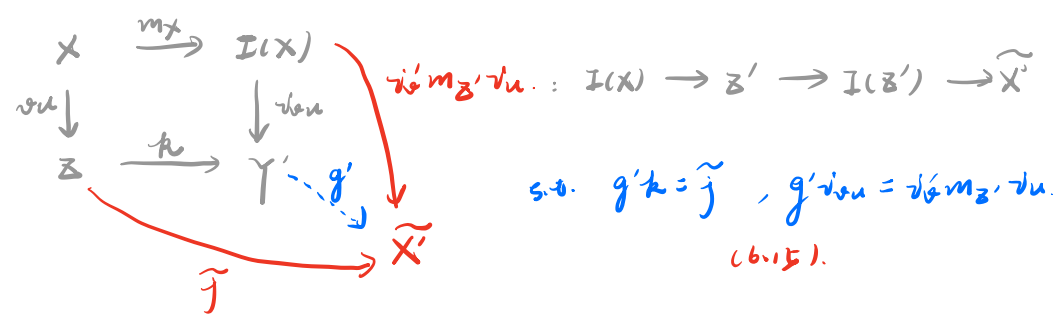
$$\begin{array}{ccccccc}
 0 & \longrightarrow & Y & \xrightarrow{m_{Z'} \circ i} & I(Z') & \xrightarrow{\pi} & M \longrightarrow 0 \\
 & & \downarrow \vartheta & & \downarrow i' & & \parallel \\
 0 & \longrightarrow & Z & \xrightarrow{\tilde{j}} & \tilde{X}' = Co & \longrightarrow & M \longrightarrow 0
 \end{array}$$

$$\begin{array}{ccc}
 & & \nearrow I(Z') \\
 Y & \longrightarrow & I(Y) \\
 \downarrow \vartheta & & \downarrow \\
 Z & \longrightarrow & X'
 \end{array}$$

lem 6.2.2 (1). \Rightarrow there are $r : X' \rightarrow \tilde{X}'$, $r' : \tilde{X}' \rightarrow X'$ s.t.

$$\begin{cases}
 r'k = \text{id}_X, & r'k' = \text{id}_{X'} \\
 r'is = i's \alpha = i's I(i), & r'j = \tilde{j} \\
 r't's = t's \alpha', & r'j = j
 \end{cases} \tag{6.14}$$

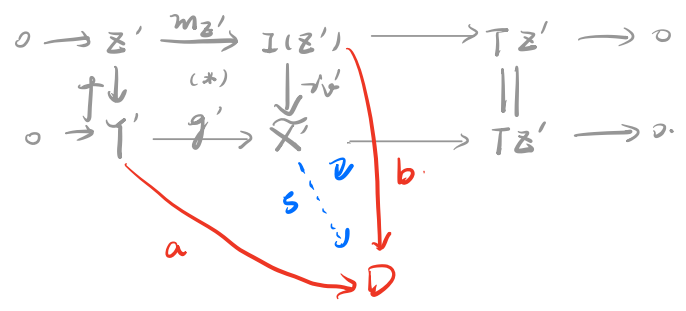
Consider



s.t. $g'k = \tilde{f}$, $g'v_X = v_X m_Z \cdot v_X$ (6.15).

$\tilde{f} v_X \stackrel{(6.13)}{=} v_X m_Z \cdot v_X \stackrel{(6.5)}{=} v_X m_Z \cdot v_X m_X$

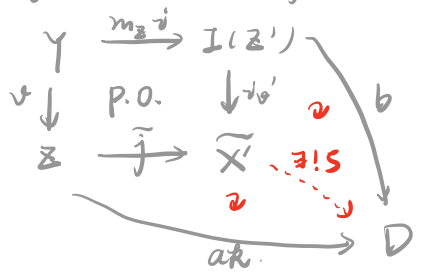
steps: (1) $Z' \rightarrow Y' \rightarrow \tilde{X} \rightarrow TZ' \in \Omega$.



\square comm. $g'f(v, -v_X) = v_X m_Z(v, -v_X)$
 $\Rightarrow (*)$ comm.

$\forall D$. If $af = bm_Z \cdot v$ find $s: \tilde{X} \rightarrow D$.

By (6.13).



$bm_Z v = af v \stackrel{(6.8)}{=} a f v$

$sg' \stackrel{?}{=} a$. $(k, -v_X): Z \oplus I(X) \rightarrow Y'$
 $sg'(k, -v_X) = \dots = a(k, -v_X) \Rightarrow sg' = a$.

$\Rightarrow Z' \xrightarrow{f} Y' \xrightarrow{g'} \tilde{X} \xrightarrow{v} TZ' \in \Omega$

(2) $Z' \xrightarrow{f} Y' \xrightarrow{v'g'} \tilde{X} \rightarrow TZ' \in \Sigma$

Let $g = v'g': Y' \rightarrow \tilde{X}$. By (6.15) & (6.14) we have:

$$\underline{gk = j} \quad \text{and} \quad g_{\text{tan}} = \alpha' m_{z'} \nu_u \quad (6.17)$$

$$\begin{array}{ccccccc} z' & \longrightarrow & Y' & \longrightarrow & \tilde{X}' & \longrightarrow & Tz' \\ \parallel & & \parallel & & \downarrow \nu' & & \parallel \\ z' & \xrightarrow{f} & Y' & \xrightarrow{g} & X' & \xrightarrow{\text{vor}} & Tz' \in \mathcal{E}. \end{array}$$

Step 4: claim $\text{vor} = (T\nu)j'$ (6.18)

consider $(j, -\nu) : z \oplus I(Y) \rightarrow X'$ surj.
 $\text{vor}(j, -\nu) = \dots = (T\nu)j' (j, -\nu).$
 $\Rightarrow \text{vor} = (T\nu)j'$

Step 5: verify $\text{vor} = (T\nu)j'$: $Y' \rightarrow$

$$(k, -\nu_u) : z \oplus I(X) \rightarrow Y'$$

By (6.5), (6.10) and (6.11)

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \xrightarrow{m_X} & I(X) & \xrightarrow{P_X} & TX \longrightarrow 0 \\ & & \downarrow u & & \downarrow \nu & & \parallel \\ 0 & \longrightarrow & Y & \xrightarrow{i} & z' & \xrightarrow{i'} & TX \longrightarrow 0 \\ & & \parallel & & \downarrow m_{z'} & & \downarrow \nu \\ 0 & \longrightarrow & Y & \xrightarrow{m_{z'} i} & I(z') & \xrightarrow{\tilde{\nu}} & M \longrightarrow 0 \\ & & \parallel & & \downarrow \alpha' & & \downarrow \beta' \\ 0 & \longrightarrow & Y & \xrightarrow{m_Y} & I(Y) & \xrightarrow{P_Y} & TY \longrightarrow 0 \end{array}$$

$$\Rightarrow \begin{array}{ccccccc} 0 & \longrightarrow & X & \xrightarrow{m_X} & I(X) & \xrightarrow{P_X} & TX \longrightarrow 0 \\ & & \downarrow u & & \downarrow \alpha' m_{z'} \nu_u & & \downarrow \beta' \nu \\ 0 & \longrightarrow & Y & \xrightarrow{m_Y} & I(Y) & \xrightarrow{P_Y} & TY \longrightarrow 0 \end{array}$$

$\downarrow I(u) \quad \downarrow T\nu$

$$\Rightarrow (\alpha' m_{z'} \nu_u - I(u)) m_X = 0.$$

there is $a : TX \rightarrow I(Y)$ s.t.

$$\alpha' m_{z'} \nu_u - I(u) = a P_X \quad (6.19)$$

$$j'g(k, -\tau u) \stackrel{(6.17)}{=} j'(j, -\tau \alpha' m_Z' \tau u) \stackrel{(6.6)}{=} (0, -P_Y \alpha' m_Z' \tau u)$$

$$(Tu)k'(k, -\tau u) \stackrel{(6.7)}{=} (0, -T(u)P_X) = (0, -P_Y I(u)).$$

Hence $\underline{j'g} - (Tu)k' = \underline{j'g} - (k, -\tau u)$

$$= (0, -P_Y (\alpha' m_Z' \tau u - I(u))) \stackrel{(6.19)}{=} (0, -P_Y \alpha P_X)$$

$$= \underline{P_Y \alpha k'} - (k, -\tau u).$$

$$\underline{j'g} = (Tu)k'.$$

#.

§ 6.3. Another explanation of d.Δ. in \mathcal{B} . Allen Hatch

prop. 6.1 Let (\mathcal{B}, S) be a Frobenius cat whose stable cat. is \mathcal{B} . Then the d.Δ. in \mathcal{B} are all induced by exact seq. in \mathcal{B} .

(i) Suppose $0 \rightarrow X \xrightarrow{u} Y \xrightarrow{v} Z \rightarrow 0 \in S$. Then $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{-w} TX \in \mathcal{E}$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \longrightarrow & 0 \\ & & \parallel & & \downarrow \sigma & & \downarrow w & & \\ 0 & \longrightarrow & X & \xrightarrow{m_X} & I(X) & \xrightarrow{P_X} & TX & \longrightarrow & 0. \end{array}$$

(6.20).

$$w', w \quad \Delta \cong \Delta'.$$

(ii) Let $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{-w'} TX' \in \mathcal{E}$, then \exists

$$\text{s.e.s. in } S: 0 \rightarrow X \xrightarrow{u} Y \xrightarrow{v} Z \rightarrow 0$$

$$\text{inducing a d.}\Delta. \text{ in } \mathcal{B}: X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{-w} TX$$

$$\text{vs isom. to } X' \rightarrow Y' \rightarrow Z' \rightarrow TX'.$$

prop. "⇒"

$$\begin{array}{ccccccc}
 0 & \longrightarrow & X & \xrightarrow{u} & Y & \xrightarrow{v} & Z \longrightarrow 0 \\
 & & m_X \downarrow & & \downarrow \begin{pmatrix} \sigma \\ \tau \end{pmatrix} & & \parallel \\
 0 & \longrightarrow & I(X) & \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & I(X) \oplus Z & \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} & Z \longrightarrow 0
 \end{array}$$

consider

$$\begin{array}{ccccccc}
 0 & \longrightarrow & X & \xrightarrow{m_X} & I(X) & \xrightarrow{P_X} & TX \longrightarrow 0 \\
 & & u \downarrow & & \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} & & \parallel \\
 0 & \longrightarrow & Y & \xrightarrow{\begin{pmatrix} \sigma \\ \tau \end{pmatrix}} & I(X) \oplus Z & \xrightarrow{\begin{pmatrix} P_X & -v \end{pmatrix}} & TX \longrightarrow 0
 \end{array}$$

⇒ $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{-v \circ u} TX \subset E.$

"⇐" ∃ standard $\Delta : (X \xrightarrow{u} Y \xrightarrow{v} Cu \xrightarrow{w} TX)$
 $\cong (X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{-w'} TX')$

try

$$\begin{array}{ccccccc}
 0 & \longrightarrow & X & \xrightarrow{m_X} & I(X) & \xrightarrow{P_X} & TX \longrightarrow 0 \\
 & & u \downarrow & & \downarrow \tilde{u} & & \parallel \\
 0 & \longrightarrow & Y & \xrightarrow{v} & Cu & \xrightarrow{w} & TX \longrightarrow 0
 \end{array}$$

⇒ s.e.s. in S :

$$\begin{array}{ccccccc}
 0 & \longrightarrow & X & \xrightarrow{\begin{pmatrix} m_X \\ u \end{pmatrix}} & I(X) \oplus Y & \xrightarrow{\begin{pmatrix} w & -\tilde{u} \end{pmatrix}} & Cu \longrightarrow 0 \\
 & & \parallel & & \downarrow \begin{pmatrix} 1 & 0 \end{pmatrix} & & \downarrow \exists -w \\
 0 & \longrightarrow & X & \xrightarrow{m_X} & I(X) & \xrightarrow{P_X} & TX \longrightarrow 0
 \end{array}$$

⇒ $X \xrightarrow{u} Y \xrightarrow{v} Cu \xrightarrow{w} TX$

#